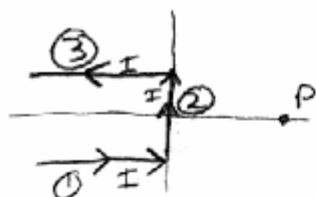


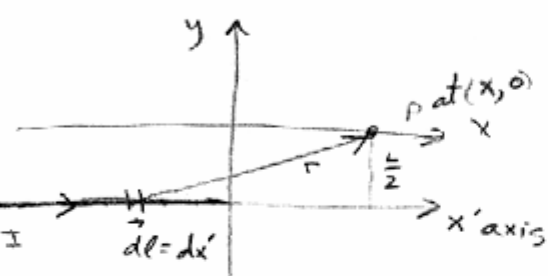
SAC 436-2

We consider one wire segment at a time.



Find \vec{B} at P due to each wire segment then add to get total \vec{B}

$$\vec{B} = \vec{B}_1 + \vec{B}_2 + \vec{B}_3$$



$$d\vec{l} = dx' \hat{i} \quad \text{Note } \hat{i} = \hat{i}' \text{ both in same direction}$$

$$\vec{r} = (-x' + x) \hat{i} + \frac{L}{2} \hat{j}$$

$$d\vec{l} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ dx' & 0 & 0 \\ -x' + x & \frac{L}{2} & 0 \end{vmatrix}$$

$$= \hat{i}(0)0 - \hat{j}(0)\frac{L}{2} + \hat{k}(0)(-x' + x) - \hat{j}(dx')0 + \hat{k}(dx')\frac{L}{2} - \hat{k}(0)(-x' + x)$$

$$d\vec{l} \times \vec{r} = \frac{L}{2} dx' \hat{k}$$

$$d\vec{B}_1 = \frac{\mu_0 I}{4\pi} \frac{d\vec{l} \times \vec{r}}{r^3}$$

$$d\vec{B}_1 = \frac{\mu_0 I}{4\pi} \frac{L}{2} \frac{dx'}{r^3} \hat{k}$$

$$r = \sqrt{(-x' + x)^2 + \left(\frac{L}{2}\right)^2}$$

All $d\vec{B}_1$'s in same direction (\hat{k})

so we can add magnitudes, then multiply by \hat{k} .

$$dB_1 = \frac{\mu_0 I L}{8\pi} \int_{-\infty}^{\infty} \frac{dx'}{[(x-x')^2 + (\frac{L}{2})^2]^{3/2}}$$

$$\text{Let } u = x - x' \quad u(-\infty) = \infty$$

$$x' = x - u \quad u(0) = x$$

$$dx' = -du$$

$$B_1 = -\frac{\mu_0 I L}{8\pi} \int_{\infty}^x \frac{du}{[u^2 + (\frac{L}{2})^2]^{3/2}}$$

using equation on formula sheet

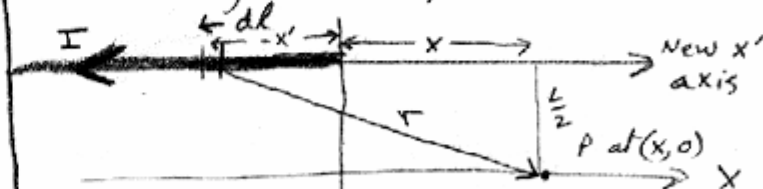
$$B_1 = -\frac{\mu_0 I L}{8\pi} \frac{1}{(\frac{L}{2})^2} \frac{u}{\sqrt{u^2 + (\frac{L}{2})^2}} \Big|_{\infty}^x$$

$$B_1 = -\frac{\mu_0 I}{2\pi L} \left\{ \frac{x}{\sqrt{x^2 + \frac{L^2}{4}}} - \frac{\infty}{\sqrt{\infty^2 + (\frac{L}{2})^2}} \right\}$$

$$B_1 = -\frac{\mu_0 I}{2\pi L} \left\{ \frac{x}{\sqrt{x^2 + \frac{L^2}{4}}} - 1 \right\}$$

$$\vec{B}_1 = \frac{\mu_0 I}{2\pi L} \left(1 - \frac{x}{\sqrt{x^2 + \frac{L^2}{4}}} \right) \hat{k}$$

We can argue that, based on the symmetry of the configuration, and the right-hand rule, $\vec{B}_3 = \vec{B}_1$, but, let's go ahead and prove that analytically by using the Biot-Savart Law to get an expression for \vec{B}_3 :



$$d\vec{l} = -dx' \hat{i} \quad (\text{Where } dx' \text{ itself is "+" is integrate from } -\infty \text{ to } 0)$$

$$r = \sqrt{(-x' + x)^2 + \left(\frac{L}{2}\right)^2}$$

$$\vec{r} = (-x' + x) \hat{i} - \frac{L}{2} \hat{j}$$

$$d\vec{l} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -dx' & 0 & 0 \\ (-x' + x) & -\frac{L}{2} & 0 \end{vmatrix}$$

$$\vec{dl} \times \vec{r} = \hat{i}(0)0 - \hat{i}(0)(-\frac{L}{2}) + \hat{j}(0)(-x'+x) - \hat{j}(-dx')0 + \hat{k}(-dx')(-\frac{L}{2}) - \hat{k}(0)(-x'+x)$$

$$\vec{dl} \times \vec{r} = \frac{L}{2} dx' \hat{k}$$

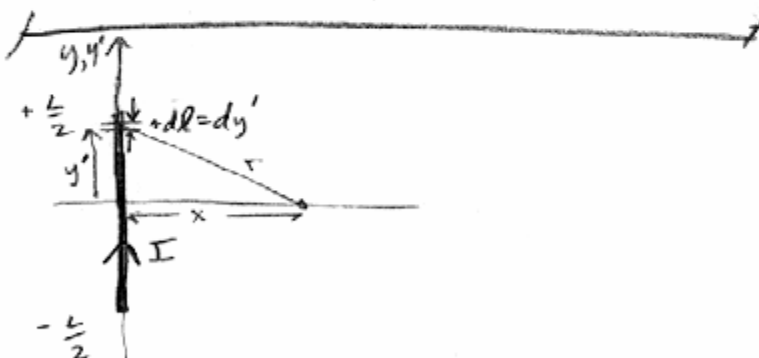
$$d\vec{B}_3 = \frac{\mu_0}{4\pi} \frac{I d\vec{dl} \times \vec{r}}{r^3}$$

$$d\vec{B}_3 = \frac{\mu_0}{4\pi} I \frac{L}{2} \frac{dx'}{r^3} \hat{k}$$

All $d\vec{B}_3$'s in same direction \hat{k}
so we can add magnitudes and
then multiply by \hat{k} .

$$dB_3 = \frac{\mu_0 I L}{8\pi} \int_{-\infty}^0 \frac{dx'}{[(x-x')^2 - (\frac{L}{2})^2]^{3/2}}$$

This is identical to the integral
we had for dB_1 . We have
already established that both
 \vec{B}_1 & \vec{B}_2 are in the $+\hat{z}$ direction,
so: $\vec{B}_3 = \vec{B}_1$



$$\vec{dl} = dy' \hat{j}$$

$$\vec{r} = x \hat{i} - y' \hat{j}$$

$$d\vec{dl} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & dy' & 0 \\ x & -y' & 0 \end{vmatrix}$$

$$\vec{dl} \times \vec{r} = \hat{i}(dy')0 - \hat{i}(0)(-y') + \hat{j}(0)x - \hat{j}(0)0 + \hat{k}(0)(y') - \hat{k}dy'x$$

$$\vec{dl} \times \vec{r} = -x dy' \hat{k}$$

$$d\vec{B}_2 = \frac{\mu_0}{4\pi} \frac{I d\vec{dl} \times \vec{r}}{r^3}$$

$$r = \sqrt{x^2 + y'^2}$$

$$d\vec{B}_2 = \frac{\mu_0}{4\pi} \frac{I (-x) dy'}{(x^2 + y'^2)^{3/2}} \hat{k}$$

All $d\vec{B}_2$'s in same direction so
we can sum magnitudes, then
multiply by \hat{k} .

$$dB_2 = -\frac{\mu_0 I x}{4\pi} \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dy'}{(x^2 + y'^2)^{3/2}}$$

This integral is on the formula
sheet.

$$B_2 = -\frac{\mu_0 I x}{4\pi} \left[\frac{1}{x^2} \sqrt{x^2 + y'^2} \right]_{-\frac{L}{2}}^{\frac{L}{2}}$$

$$B_2 = -\frac{\mu_0 I}{4\pi x} \left[\frac{(\frac{L}{2})}{\sqrt{x^2 + (\frac{L}{2})^2}} - \frac{-\frac{L}{2}}{\sqrt{x^2 + (-\frac{L}{2})^2}} \right]$$

$$B_2 = -\frac{\mu_0 I}{4\pi x} \frac{L}{\sqrt{x^2 + \frac{1}{4}L^2}}$$

Total B (which is total B_2):

$$B = B_1 + B_2 + B_3 = B_1 + B_2 + B_1$$

$$B = 2B_1 + B_2 \quad (\text{Since } B_3 = B_1)$$

$$B = 2 \left[-\frac{\mu_0 I}{2\pi L} \left(\frac{x}{\sqrt{x^2 + \frac{1}{4}L^2}} - 1 \right) \right] - \frac{\mu_0 I}{4\pi x} \frac{L}{\sqrt{x^2 + \frac{1}{4}L^2}}$$

$$B = \frac{\mu_0 I}{\pi} \left[\frac{1}{L} - \left(\frac{x}{L\sqrt{x^2 + \frac{1}{4}L^2}} + \frac{L}{4x\sqrt{x^2 + \frac{1}{4}L^2}} \right) \right]$$

$$B = \frac{\mu_0 I}{\pi} \left[\frac{1}{L} - \frac{x^2 + \frac{1}{4}L^2}{xL\sqrt{x^2 + \frac{1}{4}L^2}} \right]$$

$$B = \frac{\mu_0 I}{\pi L} \left[1 - \frac{\sqrt{x^2 + \frac{1}{4}L^2}}{x} \right]$$

$$\vec{B} = \frac{\mu_0 I}{\pi L} \left(1 - \sqrt{1 + \frac{L^2}{4x^2}} \right) \hat{k}$$